QP1 - Final 2023/2024

① a) $V_0 \to \infty$

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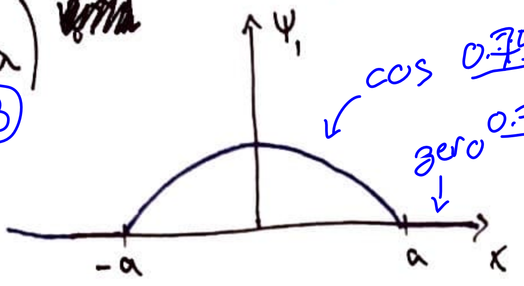

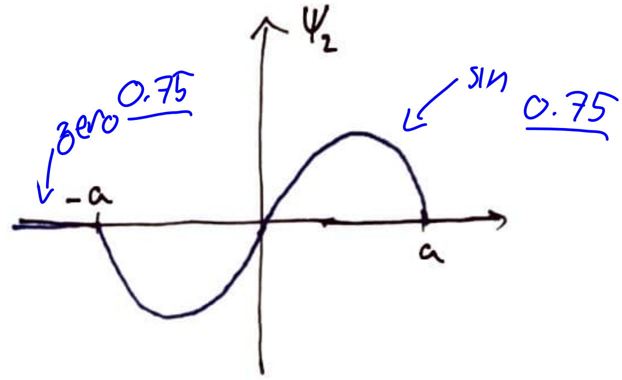

b) $H\psi = \epsilon\psi \Rightarrow -\dfrac{\hbar^2}{2m}\dfrac{d^2\psi}{dx^2} - V_0\psi = \epsilon\psi$

10

$$\frac{d^2\psi}{dx^2} = -\frac{2m(\epsilon+V_0)}{\hbar^2}\psi \Rightarrow \frac{d^2\psi}{dx^2} = -k^2\psi \quad 2 \quad k \equiv \frac{\sqrt{2m(\epsilon+V_0)}}{\hbar}$$

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

$\underline{V_0 \to \infty:}$

$$\psi(-a) = 0 \Rightarrow A\sin(ka) = B\cos(ka) \Rightarrow A = B\cot(ka)$$

$$\psi(+a) = 0 \Rightarrow B\cot(ka)\sin(ka) + B\cos(ka) = 0$$

$$2B\cos(ka) = 0$$

$$\Rightarrow \cos(ka) = 0 \Rightarrow ka = n\pi/2 \Rightarrow k = \frac{n\pi}{2a} = \frac{\sqrt{2m(\epsilon+V_0)}}{\hbar}$$

$$(\epsilon+V_0) = \frac{n^2\pi^2\hbar^2}{2m(2a)^2} \quad 2$$

For odd n:
$n = 1, 3, 5, \dots$ $\quad \psi(x) = B\cos\left(\dfrac{n\pi}{2a}x\right)$ 3

For even n:
$n = 2, 4, 6, \dots$ $\quad \psi(x) = A\sin\left(\dfrac{n\pi}{2a}x\right)$ 3

c) $\hat{D} = ex$

12

$$\langle \hat{D} \rangle = \langle \alpha(t) | \hat{D} | \alpha(t) \rangle$$

$$|\alpha(t)\rangle = a|\phi_n(t)\rangle + b|\phi_m(t)\rangle$$

$$|\alpha(t)\rangle = a e^{\frac{-i\epsilon_n t}{\hbar}} |\phi_n\rangle + b e^{\frac{-i\epsilon_m t}{\hbar}} |\phi_m\rangle \quad 2$$

$$\langle \hat{D} \rangle = \left(e^{\frac{i\epsilon_n t}{\hbar}} a^* \langle \phi_n | + e^{\frac{i\epsilon_m t}{\hbar}} b^* \langle \phi_m | \right) ex \left(e^{\frac{-i\epsilon_n t}{\hbar}} a |\phi_n\rangle + e^{\frac{-i\epsilon_m t}{\hbar}} b |\phi_m\rangle \right) = \quad 2$$

$$= |a|^2 e \langle \phi_n | x | \phi_n \rangle + e|b|^2 \langle \phi_m | x | \phi_m \rangle + a^* b \, e^{\frac{i(\epsilon_n - \epsilon_m)t}{\hbar}} e \langle \phi_n | x | \phi_m \rangle + b^* a e \, e^{\frac{i(\epsilon_m - \epsilon_n)t}{\hbar}} \langle \phi_m | x | \phi_n \rangle$$

For even n, m we have two types of integral:

$$\langle \phi_n | x | \phi_n \rangle = |A|^2 \int_{-a}^{a} x \cos^2\left(\frac{n\pi}{2a} x\right) dx = 0 \quad \text{since } x \to \text{odd function}, \ \cos^2(x) \to \text{even} \quad 3$$

$$\langle \phi_n | x | \phi_m \rangle = |A|^2 \int_{-a}^{a} x \cos\left(\frac{n\pi}{2a} x\right) \cos\left(\frac{m\pi}{2a} x\right) dx$$

Using $\cos(a)\cos(b) = \dfrac{\cos(a+b) + \cos(a-b)}{2}$

$$= |A|^2 \left[\int_{-a}^{a} x \cos\left(\frac{(n+m)\pi}{2a} x\right) dx + \int_{-a}^{a} x \cos\left(\frac{(n-m)\pi}{2a} x\right) dx \right] = 0 \quad 3$$

d) V_0 is not ∞

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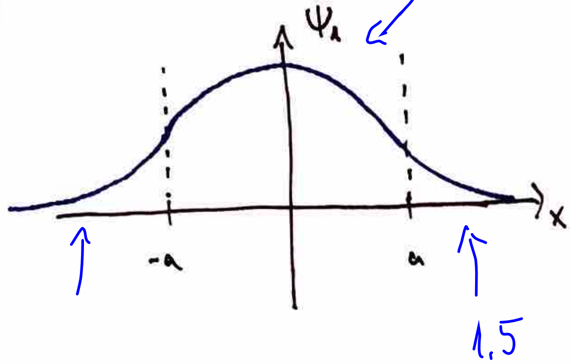

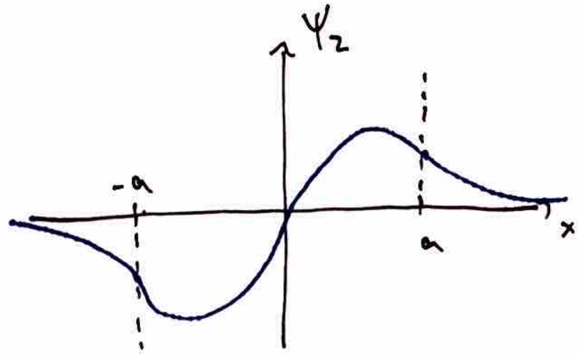

e)

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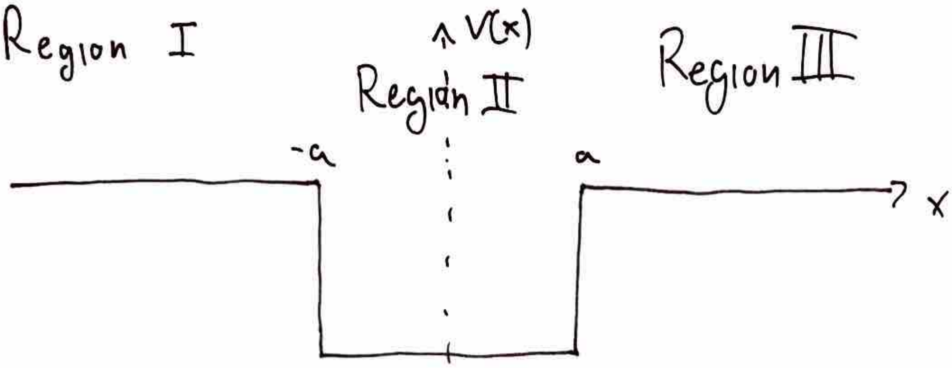


$\underline{E > 0}$

Region I:

$$\psi(x) = A e^{-ikx} + B e^{ikx} \qquad 2 \qquad k \equiv \frac{\sqrt{2mE}}{\hbar}$$

Region II:

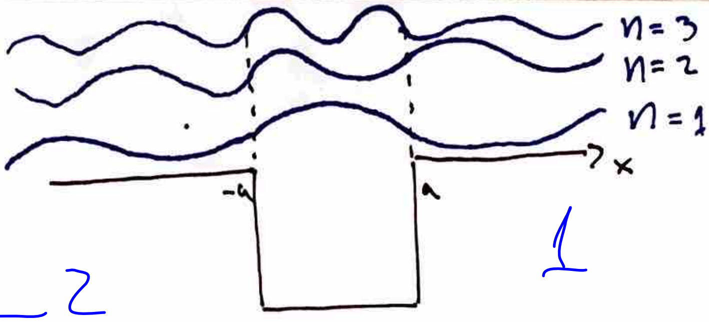
$$\psi(x) = C e^{-ilx} + D e^{ilx} \qquad 2 \qquad l \equiv \frac{\sqrt{2m(E+V_0)}}{\hbar}$$

Region III

$$\psi(x) = F e^{ikx} \qquad 2$$

f) $n\left(\frac{\lambda}{2}\right) = 2a$

6



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Transmission = 1

2

$$\lambda = \frac{h}{p} = \frac{2\pi\hbar}{\hbar k} = \frac{2\pi}{k} \Rightarrow n\left(\frac{\lambda}{2}\right) = n\cdot\left(\frac{2\pi}{2k}\right) = \frac{n\pi}{k} = 2a \Rightarrow k = \frac{n\pi}{2a}$$

$$k = \frac{\sqrt{2m(\epsilon + V_0)}}{\hbar} = \frac{n\pi}{2a} \Rightarrow \underbrace{(\epsilon + V_0) = \frac{n^2\pi^2\hbar^2}{2m(2a)^2}}_{3}$$

Same as inf. sq. well!

(2) [30] $H = -\gamma \vec{S}\cdot\vec{B}$ $\qquad \vec{S} = (S_x, S_y, S_z)$

5 a) $\vec{B} = B_0\hat{z} \Rightarrow H = -\gamma S_z B_0 \qquad S_z = \frac{\hbar}{2}\sigma_z$

$$H = -\gamma B_0 \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad 2 \qquad \mathcal{E} \equiv -\gamma B_0 \frac{\hbar}{2}$$

$$H = \begin{pmatrix} +\mathcal{E} & 0 \\ 0 & -\mathcal{E} \end{pmatrix}$$

$$\begin{vmatrix} \mathcal{E}-\epsilon & 0 \\ 0 & -\mathcal{E}-\epsilon \end{vmatrix} = 0 \Rightarrow (\mathcal{E}-\epsilon)(\mathcal{E}+\epsilon) = 0 \quad \mathcal{E}^2 - \epsilon^2 = 0 \quad \epsilon = \pm\mathcal{E}$$

$$\epsilon_+ = -\gamma B_0 \frac{\hbar}{2} \qquad \epsilon_- = \gamma B_0 \frac{\hbar}{2} \quad 2$$

$$\begin{pmatrix} \mathcal{E} & 0 \\ 0 & -\mathcal{E} \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \epsilon_\pm \begin{pmatrix} a \\ b \end{pmatrix} = \pm\mathcal{E}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} \mathcal{E}a = \pm\mathcal{E}a \\ -\mathcal{E}b = \pm\mathcal{E}b \end{matrix} \Rightarrow \begin{matrix} |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ |-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix} \quad 1$$

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10 b) $S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad |+(t)\rangle_z = ?$

$$\begin{vmatrix} -\epsilon & \hbar/2 \\ \hbar/2 & -\epsilon \end{vmatrix} = \epsilon^2 - \left(\frac{\hbar}{2}\right)^2 = 0 \Rightarrow \epsilon_\pm = \pm\frac{\hbar}{2} \quad 2$$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} a = \pm b \\ b = \pm a \end{matrix} \Rightarrow \text{eigenvectors} \begin{cases} |+\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ |-\rangle_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases} \quad 2$$

$$|+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] \Rightarrow |+\rangle_z = \frac{1}{\sqrt{2}}\left(|+\rangle_x + |-\rangle_x\right) \quad 2$$

$$|+(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\epsilon_+ t/\hbar}|+\rangle_x + e^{-i\epsilon_- t/\hbar}|-\rangle_x\right] \quad 2 \qquad H = -\gamma B_0 S_x \Rightarrow \epsilon_\pm = \mp\gamma B_0\frac{\hbar}{2}$$

$$|+(t)\rangle = \frac{1}{\sqrt{2}}\left[e^{i\frac{\gamma B_0}{2}t}|+\rangle_x + e^{-i\frac{\gamma B_0}{2}t}|-\rangle_x\right] \quad 2$$

7 c) $S_z|+\rangle_x = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$

$$S_z|+\rangle_x = \frac{\hbar}{2}|-\rangle_x \qquad 3.5$$

$$S_z|-\rangle_x = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \frac{\hbar}{2}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$S_z|-\rangle_x = \frac{\hbar}{2}|+\rangle_x \qquad 3.5$$

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8 d) $\langle S_x\rangle = {}_z\langle +(t)|S_x|+(t)\rangle_z = \left[\frac{1}{\sqrt{2}}\left(e^{-i\frac{\gamma B_0 t}{2}}{}_x\langle +| + e^{\frac{i\gamma B_0 t}{2}}{}_x\langle -|\right)\right] S_x \left[\frac{1}{\sqrt{2}}\left(e^{\frac{i\gamma B_0 t}{2}}|+\rangle_x + e^{-\frac{i\gamma B_0 t}{2}}|-\rangle_x\right)\right]$

$$= \frac{1}{2}\left[{}_x\langle +|S_x|+\rangle_x + {}_x\langle -|S_x|-\rangle_x + e^{-i\gamma B_0 t}{}_x\langle +|S_x|-\rangle_x + e^{i\gamma B_0 t}{}_x\langle -|S_x|+\rangle_x\right] =$$

$$= \frac{1}{2}\left[\frac{\hbar}{2}\underbrace{{}_x\langle +|+\rangle_x}_{1} + \left(-\frac{\hbar}{2}\right)\underbrace{{}_x\langle -|-\rangle_x}_{1} + e^{-i\gamma B_0 t}\left(-\frac{\hbar}{2}\right)\underbrace{{}_x\langle +|-\rangle_x}_{0} + e^{i\gamma B_0 t}\left(\frac{\hbar}{2}\right)\underbrace{{}_x\langle -|+\rangle_x}_{0}\right]$$

$$= \frac{1}{2}\left[\frac{\hbar}{2} - \frac{\hbar}{2}\right] \Rightarrow \boxed{\langle S_x(t)\rangle = 0} \qquad 4$$

$$\langle S_z\rangle = \left[\frac{1}{\sqrt{2}}\left(e^{-i\frac{\gamma B_0 t}{2}}{}_x\langle +| + e^{\frac{i\gamma B_0 t}{2}}{}_x\langle -|\right)\right] S_z\left[\frac{1}{\sqrt{2}}\left(e^{\frac{i\gamma B_0 t}{2}}|+\rangle_x + e^{-\frac{i\gamma B_0 t}{2}}|-\rangle_x\right)\right]$$

$$= \frac{1}{2}\left[{}_x\langle +|S_z|+\rangle_x + {}_x\langle -|S_z|-\rangle_x + e^{-i\gamma B_0 t}{}_x\langle +|S_z|-\rangle_x + e^{i\gamma B_0 t}{}_x\langle -|S_z|+\rangle_x\right]$$

$$= \frac{1}{2}\left[\frac{\hbar}{2}{}_x\langle +|-\rangle_x + \frac{\hbar}{2}{}_x\langle -|+\rangle_x + e^{-i\gamma B_0 t}\frac{\hbar}{2}{}_x\langle +|+\rangle_x + e^{i\gamma B_0 t}\frac{\hbar}{2}{}_x\langle -|-\rangle_x\right]$$

$$= \frac{\hbar}{2}\left[\frac{e^{i\gamma B_0 t} + e^{-i\gamma B_0 t}}{2}\right] \Rightarrow \boxed{\langle S_z\rangle = \frac{\hbar}{2}\cos(\gamma B_0 t)} \qquad 4$$

(3) a) $a_+|\psi_n\rangle = \sqrt{n+1}\,\psi_{n+1}$ $\qquad a_-|\psi_n\rangle = \sqrt{n}\,\psi_{n-1}$

3 $\qquad$ 1.5

$$a_+a_-|\psi_n\rangle = \sqrt{n}\,a_+|\psi_{n-1}\rangle = \sqrt{n}\sqrt{n-1+1}\,|\psi_n\rangle \Rightarrow a_+a_-|\psi_n\rangle = n|\psi_n\rangle$$

$$a_-a_+|\psi_n\rangle = \sqrt{n+1}\,a_-|\psi_{n+1}\rangle = \sqrt{n+1}\sqrt{n+1}\,|\psi_n\rangle \Rightarrow a_-a_+|\psi_n\rangle = (n+1)|\psi_n\rangle$$

1.5

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b) $\langle x\rangle = \langle\psi_n|\hat{x}|\psi_n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\,\langle\psi_n|(a_++a_-)|\psi_n\rangle$ 1

8

$$\langle x\rangle = 0$$

$\langle p\rangle = \langle\psi_n|\hat{p}|\psi_n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\,\langle\psi_n|(a_+-a_-)|\psi_n\rangle$ 1

$$\langle p\rangle = 0$$

$x^2 = \frac{\hbar}{2m\omega}(a_++a_-)(a_++a_-) \Rightarrow x^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right)$ 1

$p^2 = -\frac{\hbar m\omega}{2}(a_+-a_-)(a_+-a_-) \Rightarrow p^2 = -\frac{\hbar m\omega}{2}\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right)$ 1

$$\langle x^2\rangle = \frac{\hbar}{2m\omega}\langle\psi_n|\left(a_+^2 + a_+a_- + a_-a_+ + a_-^2\right)|\psi_n\rangle$$

$= \frac{\hbar}{2m\omega}\left[0 + n + (n+1) + 0\right] \Rightarrow \langle x^2\rangle = \frac{\hbar}{2m\omega}(2n+1)$ 1

$$\langle p^2\rangle = -\frac{\hbar m\omega}{2}\langle\psi_n|\left(a_+^2 - a_+a_- - a_-a_+ + a_-^2\right)|\psi_n\rangle$$

$= -\frac{\hbar m\omega}{2}\left[0 - n - (n+1) + 0\right] \Rightarrow \langle p^2\rangle = \frac{\hbar m\omega}{2}(2n+1)$ 1

$$\sigma_x^2\,\sigma_p^2 = \left(\langle x^2\rangle - \langle x\rangle^2\right)\left(\langle p^2\rangle - \langle p\rangle^2\right)$$

$$= \left[\frac{\hbar}{2m\omega}(2n+1)\right]\left[\frac{\hbar m\omega}{2}(2n+1)\right] = \left(\frac{\hbar}{2}\right)^2 (2n+1)^2$$

$$\Rightarrow \sigma_x\,\sigma_p = \frac{\hbar}{2}(2n+1) \geq \frac{\hbar}{2} \qquad 2$$

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7 c) $a_-\psi_0 = 0 \Rightarrow \dfrac{1}{\sqrt{2\hbar m\omega}}\left(i\hat{p} + m\omega x\right)\psi_0 = 0$ 1

$$\hat{p} = -i\hbar\frac{d}{dx} \Rightarrow \frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0 = 0 \qquad 1$$

$$\hbar\frac{d\psi_0}{dx} + m\omega x\psi_0 = 0 \overset{1}{\Rightarrow} \frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\psi_0 \overset{1}{\Rightarrow} \int\frac{d\psi_0}{\psi_0} = -\frac{m\omega}{\hbar}\int x\,dx \quad 1$$

$$\ln\psi_0 = -\frac{m\omega}{\hbar}\frac{x^2}{2} + C \quad 1 \quad \Rightarrow \boxed{\psi_0(x) = A\,e^{-\frac{m\omega}{2\hbar}x^2}} \quad 1$$

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d) $a_+\psi_0 = \dfrac{1}{\sqrt{2\hbar m\omega}}\left(-\hbar\dfrac{d}{dx} + m\omega x\right)A\,e^{-\frac{m\omega}{2\hbar}x^2}$ 1

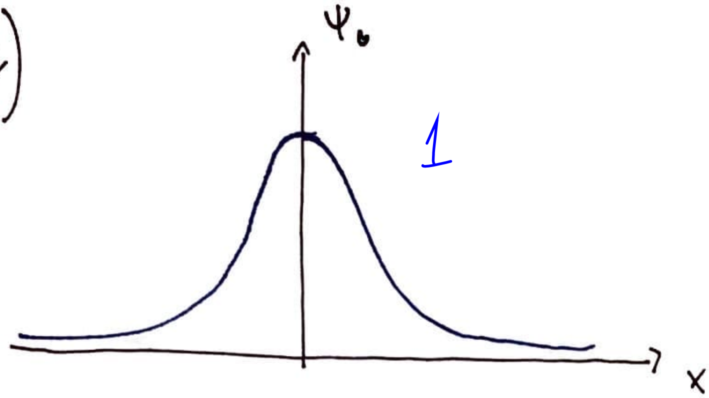
7

$$= \frac{A}{\sqrt{2\hbar m\omega}}\left[-\hbar\left(-\frac{m\omega}{2\hbar}2x\right)e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x\,e^{-\frac{m\omega}{2\hbar}x^2}\right] \quad 1$$

$$= \frac{A}{\sqrt{2\hbar m\omega}}\left[m\omega x\,e^{-\frac{m\omega}{2\hbar}x^2} + m\omega x\,e^{-\frac{m\omega}{2\hbar}x^2}\right] \quad 1$$

$$\boxed{\psi_1 = 2A\sqrt{\frac{m\omega}{2\hbar}}\;x\,e^{-\frac{m\omega}{2\hbar}x^2}} \quad 1$$

e) 2



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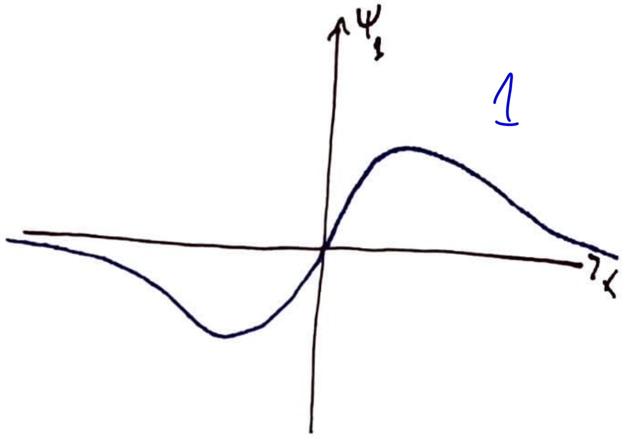

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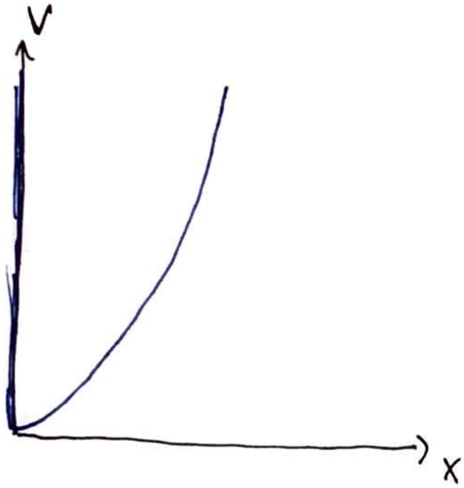
f) 3



$\psi = 0$ for $x = 0$

only odd n's survive

Ground state $\Rightarrow \psi_1$